Blue eyed Villagers

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1 Puzzle

For this puzzle, we go to that favourite retreat of mathematicians, an island full of perfect logicians. The island we go to has a few interesting properties. Firstly, the people there are all have either brown or blue eyes. Secondly, there is a ferry that leaves the island every day at 6pm, but the ferrymaster is a rather mean old man. When he arrives he asks everyone on the dock what their eye colour is. If they get it right, he takes them to the nearby island of eternal happiness, but if they get it wrong he kills them instantly. As such, the inhabitants are rather nervous of heading down to the ferryport, and won't do so unless they are completely sure that they know what their eye colour is. Sadly for them there are no reflective surfaces anywhere on the island, so they have no way of seeing their eye colour, and they have no language or way of communicating with each other - so they need to work out their eye colour using logical deduction along.

One day the local sage comes down into the island's one village as they are having their lunch. Lunch is a popular affair on the island, and every day they all gather at noon in the middle of the village for it. He loudly informs the assembled crowd.

> "I can see someone with blue eyes"

A few days pass, no changes on the island. No-one goes to the ferryport, and things continue as normal. However, exactly 100 days after he makes his statement, there is a large crowd at the ferryport of people all claiming to have blue eyes (and naturally they are all correct).

The question is, given that there are 200 villagers on the island (the sage is assumed to be happy there and so isn't included), can you work out how many people were waiting at the dock? Would your answer change if there were 300 people on the island? Would it change if there were people with blue, green and brown eyes?

2 Discussion

This is a surprisingly popular form of puzzle, and there are a few things which you are expected to assume when told everyone is a perfect logician. Firstly, they won't make any guesses (they can make *assumptions* as part of logical reasoning, but they won't act on them until they are either confirmed or refuted). So if people were waiting at the dock, it wasn't because they *suspected* that they had blue eyes, but because they *knew*.

More to the point, they knew on day 100 and didn't know on day 99. So you have to consider what extra information they would have had on day 100 versus 99. Because this is a mathematical puzzle not a trick question it isn't going to be something like "an explorer arrived on the island and sold them all mirrors". Don't forget that they always all turn up for lunch - provided that they are still on the island anyway!

If you are struggling with this puzzle, it's often better to start looking at simpler examples. What happens if there is only one villager? How about if there are two - and separately look at the case where they have the same, or differing eye colours? Then work up to three, looking at the various cases. From there you should start to see a pattern.

Finally, I am aware of the logical inconsistency with them having no language (and thus being unable to tell each other what colour their eyes are), yet the sage also being able to speak to them all. You can ignore that. If you prefer, let them all be mute, or assume there are nearby guards who will kill anyone who breaks the vow of silence. No-one ever said these puzzles had to be sensible...

3 Solution

Initially the puzzle looks impossible - so as is normal for mathematicians, we try to solve an easier one.

Let's initially assume there is only one person in the village. Of course, he has no way of knowing what colour eyes he has. However, once the sage makes his announcement that he can see at least one person with blue eyes, our villager can obviously work out that it's him, and so heads down to the ferry port that evening. So with only one, necessarily blue-eyed, villager, he would present himself at the ferry port on day one.

We then try looking at a village with 2 people. Let's first of all assume that we have 1 blue-eyed and 1 brown-eyed. On day 1 the sage comes down and makes his claim that he can see at least one blue-eyed person. The brown-eyed chap is rather non-plussed, but the blue-eyed one can reason as follows:

1) There is at least 1 blue eyed person in this village
2) It's not the other person
3) So it must be me

And so he heads down to the docks that evening, knowing he has blue eyes.

How about if we have 2 blue-eyed villagers? In that case, on day one, both would hear the sage, and not be able to know if they themselves have blue or brown eyes (the sage could be talking about the other chap after all).

However, on day 2 they would meet for lunch. And each would notice that the other hasn't left. This would enable them to reason as follows:

1) If this village only had 1 blue-eyed person, he would have left on day 1
2) No-one has left
3) Therefore we must have more than 1 blue-eyed person. I can only see 1, so we must have 2, with the second being me!

And both of them would present themselves at the docks that evening.

We can continue with this sort of argument to a village with 3 people (looking individually at the cases for 1, 2 or 3 having blue eyes) and then 4 etc. Very soon a pattern emerges, which is that regardless of how many brown-eyed people there are, if there are n blue-eyed people they all head down to the docks on day n. And you can check this using induction.

First of all, we define our induction hypothesis: namely that if there are n blue-eyed people in the village they will be able to conclude that they have blue eyes on the n^{th} day.

This is easily verified for $n = 1$ - if there is only 1, regardless of how many brown-eyed people there are, when the sage says he can see at least 1 blue-eyed person, he would look around and not be able to see any others, so know it is him. So our claim is true for $n = 1$.

(You could similarly explicitly go through the logic for $n = 2$ if it makes you feel better, it's very similar)

Now assume that it is true up to $n = k$ i.e. for all $n \leq k$ if we have n blue-eyed vilagers they will work out their eye-colour on the n^{th} day. Assume now that we have $k + 1$ blue-eyed villagers. Each is a perfect logician, so has worked out that if we have up to k blue-eyed villagers they would leave on the appropriate day. Each of them that has blue eyes can see k blue-eyed vilagers, so would expect them to leave on day k. However, when day k comes and goes, the next day at lunch they note that no-one has left. Thus they know there are $k + 1$ blue-eyed villagers, including themselves, and so go on day $k + 1$. Thus our induction works, and is true forall n.

Now to answer the questions as phrased. We know given that they went on the 100^{th} day that there must be 100 blue-eyed villagers. Note that this doesn't change at all if we have more or fewer brown-eyed villagers - sadly they get left behind. So even if there are 300, or 400, or 1,000,000 (provided they can all see each others eyes and count them quickly enough, which gets hard with 1,000,000 people) we get the same answer.

Adding more eye colours to the mix only makes one rather sad difference. If everyone else has brown eyes, then they can be very lazy. As they saw all the blue-eyed ones leave on the same day, they can therefore conclude that they have brown eyes by default and head down the next day. However, if any of them might have another colour (such as green) then they can never leave, because they cannot tell at all which they are, all they know is that they aren't blue. Unless of course the sage comes back.

There is another interesting side to the puzzle - is the sage necessary? Obviously if there is only 1 blue-eyed villager he is, but if there is more than one then everyone can see that there is at least one blue-eyed villager - so what does the sage add to the equation? There is a school of thought that all he does is provide a convenient starting point to begin counting from: this isn't true. If you try the same problem with the ferryman only starting work on a specific day (also providing a starting point) then no-one can ever conclude what colour their eyes are.

The reason for this is that they have assymetric information: Let's take the case of there being 2 blue-eyed villagers only in our village. Each knows there is at least one blue-eyed villager. However, they *don't know* that the other villager also knows this - if our villagers were called Alf and Bert, Alf knows that the village has at least one pair of blue eyes, but doesn't know if Bert (who can only see Alf) knows that the village has at least one pair of blue eyes, because he doesn't know what colour eyes Bert can see. When the sage comes down, the assures both that they know that the village has some blue eyes in it, but also that they know that the other knows this. If this is giving you a bit of headache - don't worry, it's quite a subtle but important difference.

Fine for 2, but we have 100 people with blue eyes. Surely they can all assume that everyone knows that everyone knows that everyone knows ... there are some blue eyes? Sadly no - the explanation gets quite complicated (and essentially consists of A not knowing that B knows that C knows that D ...), but adding more blue-eyed villagers means they need to be able to assume more knowledge to work it out, and eventually it breaks down. You need the sage for everyone to have complete knowledge of what everyone knows.