Chessboard Coverings

Summer 2009

Contents

1 Puzzle

A slightly easier one this week, in two parts. Let's assume you have dominoes which are conveniently exactly the size of 2 chessboard squares. You can easily cover one corner of the board, there are several solutions (how many?), Figure 1 shows one:

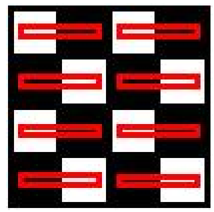

Figure 1: Covering a corner of the chessboard

How about if we remove two opposing corners? Can you cover the truncated corner chessboard in Figure 2?

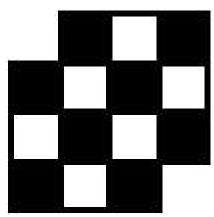

Figure 2: Truncated corner chessboard

How about the truncated full chessboard as in Figure 3?

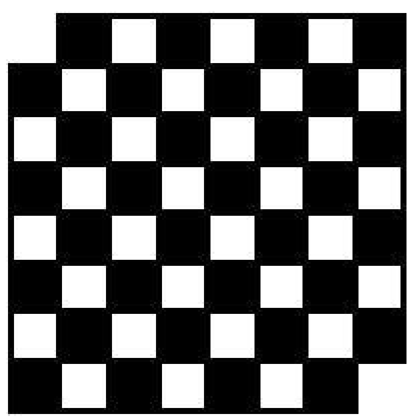

Figure 3: Truncated full chessboard

2 Discussion

There are quite a few different ways of counting the coverings for the corner chessboard, that I'll leave you to work out yourself. Don't forget to include reflections and rotations!

When it comes to covering the truncated corner chessboard, trial and error (or trial and improvement if you want to be clever about it) will probably get you there.

For the larger one, although it is possible to do it that way, it will take an awfully long time. There is a quicker way, which relies on the way the diagrams are drawn. I appreciate this isn't much of a hint, but you don't want this to be too easy, do you?

3 Solution

I'll actually come back to the problem of how many ways there are of covering the complete corner at the end - it's a slightly involved counting problem, and distracts from the main point of this week's puzzle.

So, can we cover the corner chessboard with two opposing corners cut off? As you might have guessed, the answer is no. There are two typical approaches to adopt, the first is a simple brute force approach as shown in Figure 4:

First off, assume we place a domino in position ① to cover the bottom left corner. This forces us to place ② to cover the square above, which forces ③ etc etc. You end up placing the first 6 dominos as shown, and then have no way of covering the last 2 squares.

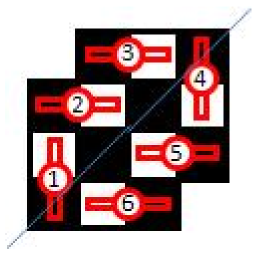

Figure 4: Attempting to cover a truncated corner chessboard

You could equally place domino ① horizontally - you'll quickly discover that you get a similar problem, because it's essentially the same pattern, just reflected in the diagonal axis labelled in blue.

However, when we come to the full chessboard with two opposing corners removed, it turns into a bit of a nightmare. The shear number of possibilities makes this approach pretty impractical. Instead, we have to use the alternative approach, which is much more cunning.

First off, the fact that we are using a chessboard not a blank grid is actually significant. Note that the chessboard has alternating black and white squares, with each black square surrounded by 4 white squares, and vica versa. You can also quite easily see that each domino covers one black and one white square, regardless of where it is placed.

So you can only cover grids with equal numbers of black and white squares. And if you take off two opposing corners then you are removing two of the same colour, leaving mismatched numbers of coloured squares, and so it is impossible. Obviously this is much quicker, but best of all it's a nice general approach - without having to even look at any I can say directly that no chessboard of any size with the opposing corners removed can be covered.

Now we come to the question I skipped earlier, and which I now slightly regret having posed. Initially the question of how to count the solutions sounds very difficult, but the trick is to break it down into several categories, count each of them and then add them up. Sounds pretty straightforward?

The categories I am going to use are to look at how the 2x2 corners interact with each other. First of all, we look at the cases where they don't overlap - i.e. each corner is independently covered with 2 dominos, either going horizontally or vertically. As these are independent each can cycle through each of their 2 alternatives, and there are 4 of them, so we have $2^4 = 16$ solutions.

Next, we want to look at ones which do interact. First of all, just look at the ones where the top left interacts with the bottom left - and then rotate these where possible to get the other solutions.

The top left can interact with the bottom left in three ways - as shown in figure 5 - either on the left, on the right, or over both of the parts of the edge. Because we require in each case that that is the only link between the top left and bottom left, in the first 2 cases there is only one solution - each being a rotation of the other.

In the third case, we have 5 ways of filling the right hand side. The first 4 are those in which the corners don't interact - each could go horizontally or vertically, so we get $2^2 = 4$ solutions. And each of these gives rise to 4 more solutions through rotations, so we have 16 of those.

Finally we have the solution in which the two sides do interact - and we have only one of these, the same as the left hand side reflected over. And this only gives rise to 2 solutions through rotations - again as rotating by 180° gives us back our initial solution.

So in total how many do we have? 16 from the independent corners, and then $2 + 16 + 2$ from those where corners do interact, so in total we have 36. And for anyone interested, here they all are:

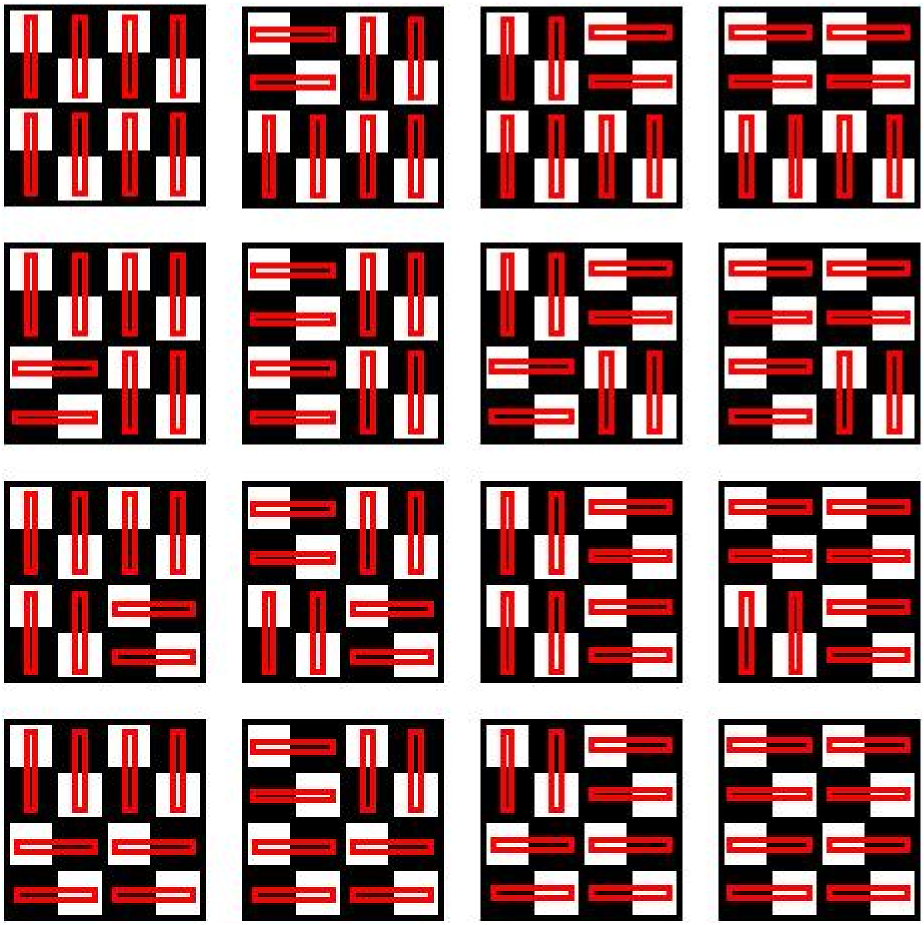

Figure 5: 16 Solutions with independent corners

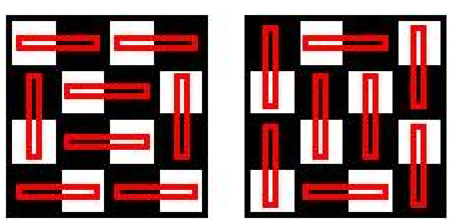

Figure 6: 2 Solutions with a single overlap

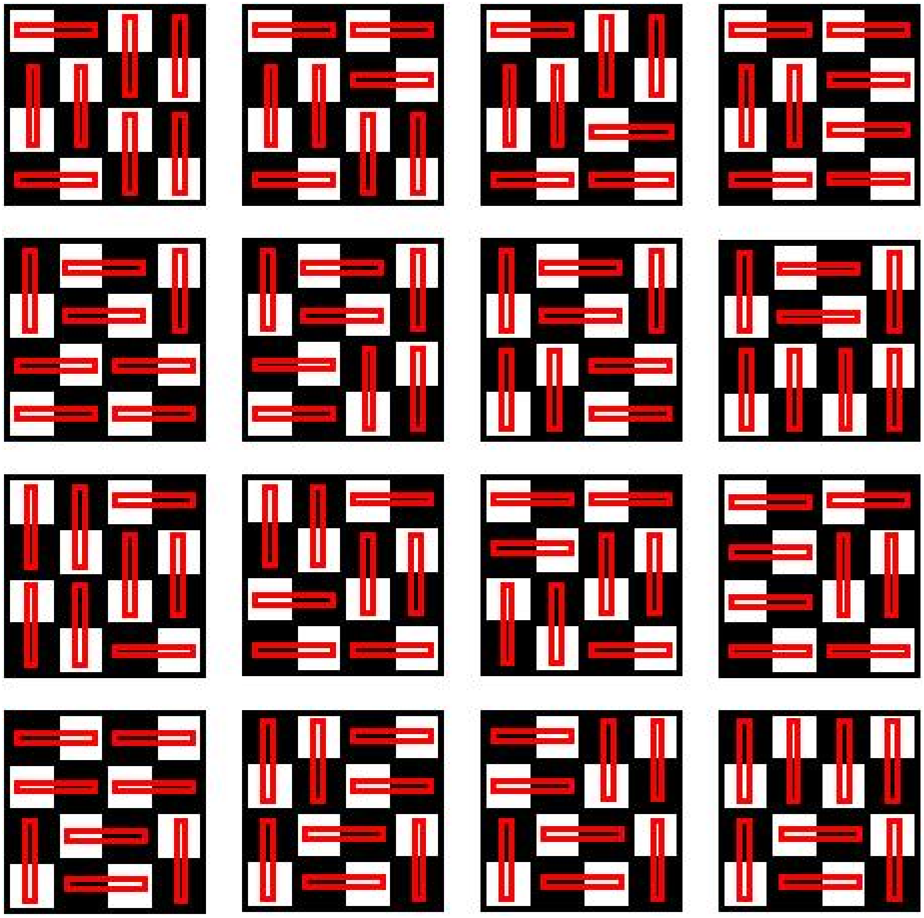

Figure 7: 16 Solutions with 2 overlaps and independent corners

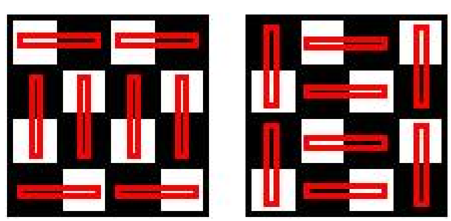

Figure 8: 2 solutions with 2 overlaps, reflected on the other side