Poker

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1 Puzzle

This puzzle is based on one presented by Martin Gardner in "Mathematical Puzzles and Diversions" (ISBN 0140136355)

Assume we have two perfect card sharks playing poker. Each has complete control of the deck and deals their own cards, so when they are passed the deck not only can they control which cards they receive but they can also do so knowing which have already been removed (and thus what their opponent holds). Assuming they play a single-draw game of stud poker (i.e. as follows)

- The first player draws a hand by picking any 5 cards from the deck
- The second player does the same from the remaining 47 cards
- The first player then discards as many of his cards as he likes and deals himself new cards from the remaining deck
- The second can then also discard and re-draw from the main deck, not including the cards discarded by the first player

What hand should the first player deal himself to be guaranteed of victory? How many such hands are there?

2 Discussion

First off it is worthwhile recalling the poker hands and their relative value:

Straight Flush - A straight flush is a poker hand which contains five cards in sequence, all of the same suit, such as Q ♣ J ♣ 10 ♣ 9 ♣ 8 ♣. Two such hands are compared by their highest card - note that although Aces can be played low this would then render them as having a value of 1, and so no longer being considered the highest card. An Ace-high straight flush such as A ♠ K ♠ Q ♠ J ♠ 10 ♠ is known as a *Royal Flush* and is the highest ranking standard poker hand.

Four of a kind - Also known as *quads* is a poker hand such as 9 ♠ 9 ♡ 9 ♢ 9 ♣ J ♡ which contains four cards of one rank and an unmatched card of another rank. Higher ranking quads defeat lower ranking quads. In the case that two players have equal matching quads (e.g. when playing Texas Hold'Em) the unmatched card acts as a kicker (so 9 ♠ 9 ♡ 9 ♢ 9 ♣ J ♡ beats 9 ♠ 9 ♡ 9 ♢ 9 ♣ 3 ♣)

Full House - A full house is a hand such as 3 ♣ 3 ♠ 3 ♢ 6 ♣ 6 ♡ which contains three matching cards of one rank and two matching cards of another rank. Between two full houses the one with the higher ranking set of three wins, so 7 ♠ 7 ♡ 7 ♢ 4 ♠ 4 ♣ defeats 6 ♠ 6 ♡ 6 ♢ A ♠ A ♣.

Flush - A flush is a poker hand such as Q ♣ 10 ♣ 7 ♣ 6 ♣ 4 ♣ which contains five cards of the same suit, not in rank sequence. Two flushes are compared as if they were high card hands, the highest ranking card of each is compared to determine the winner. If both hands have the same highest card, then the second-highest ranking card is compared, and so on until a difference is found.

Straight - A straight is a poker hand such as Q ♣ J ♠ 10 ♠ 9 ♡ 8 ♡ which contains five cards of sequential rank but in more than one suit. Two straights are ranked by comparing the highest card of each. Aces may be either played high or low, but not both (so 3 ♣ 2 ♢ A ♡ K ♠ Q ♣ is not a straight).

Three of a Kind - A three of a kind is a poker hand such as 2 ♢ 2 ♠ 2 ♣ K ♠ 6 ♡ which contains three cards of the same rank, plus two unmatched cards. Higher-valued three of a kind defeat lower-valued three of a kind, so Q ♠ Q ♡ Q ♢ 7 ♠ 4 ♣ defeats J ♠ J ♢ J ♣ A ♢ K ♣. If two hands contain threes of a kind of the same value the kickers are sequentially compared to break the tie.

Two Pair - A poker hand such as J ♡ J ♣ 4 ♠ 4 ♣ 9 ♡ which contains two cards of the same rank, plus two cards of another rank (that match each other but not the first pair), plus one unmatched card, is called two pair. To compare two hands each containing two pair, the higher ranking pair of each is first compared, and the higher pair wins (so 10 ♠ 10 ♣ 3 ♡ 3 ♣ 4

♠ defeats 9 ♡ 9 ♣ 8 ♠ 8 ♣ A ♠). If both hands have the same "top pair" then the second pair of each is compared, such that 10 ♠ 10 ♣ 8 ♡ 8 ♣ 4 ♠ defeats 10 ♡ 10 ♢ 3 ♠ 3 ♡ A ♣. Finally, if both hands have the same two pairs, the kicker determines the winner.

One Pair - One pair is a poker hand such as 4 ♡ 4 ♠ K ♠ 10 ♢ 5 ♠ which contains two cards of the same rank plus three other unmatched cards. Higher ranking pairs defeat lower ranking pairs; if two hands have the same pair, the non-paired cards are sequentially compared to determine the winner.

High Card - A high card or "no-pair" hand is a poker hand such as K ♡ J ♣ 8 ♣ 7 ♢ 3 ♠, in which no two cards have the same rank, the five cards are not in sequence and the five cards are not all the same suit. High card hands are ranked by comparing the highest ranking card. If those two are equal, then the next highest ranking card from each hand is compared, and so on until a difference is found.

We should also note that the puzzle calls for some specific requirements. The first player should be able to pick 5 cards on his first turn as to guarantee a *win* - i.e. whatever the second takes the first player can be sure he can make a better hand. This may involve him changing his hand on his second turn or not, regardless the second player should not be able to force a draw or win himself.

Note that two equally high royal flushes in different suits would then result in a draw.

Head to the next page for the solution.

3 Solution

Instincticely one is drawn to the most powerful hand available, a royal flush, so one would assume that the first player takes one e.g.

$$A\,\spadesuit,\ K\,\spadesuit,\ Q\,\spadesuit,\ J\,\spadesuit,\ 10\spadesuit$$

This doesn't work however, as he cannot precent the second player from dealing himself a royal flush in another suit, which the second player could just sit on, giving him at least a draw and preventing the first player from winning.

It is thus clear that the first player must take enough cards on his first turn to prevent his opponent from being able to pick out a royal flush. Further, given that straight flushes are the strongest hands, and that it is impossible to stop his opponent from forming one by removing only 5 cards, the first player must leave himself able to form a higher straight flush than his opponent on his second turn.

In order to achieve the first condition (preventing his opponent from forming a royal flush) he will need to pick a royal from each of the four suits. The most obvious option would be to pick the highest ones, in the hope that that would mean he could make the highest possible straight flush on his next turn. So his next logical option would be

$$A\,\clubsuit,\ A\,\diamondsuit,\ A\,\heartsuit,\ A\,\spadesuit,\ K\,\spadesuit$$

(The fifth card isn't enormously important, for the sake of illustration I've chosen a royal, but it doesn't really matter).

However, this is easily countered. If the second player takes lower royals in all four suits (for example, continuing on from the first player taking all the aces, he could take all four queens), he prevents the first player from turning any of his initial hand into a royal flush. Furthermore, as both players are ultimately seeking straight flushes, the first player is forced to discard his entire initial hand and make up a straight flush from what is left. Necessarily any such straight flush is strictly limited above by the second player's card - all he has to do is turn any royal card in his hand into a straight flush to win - so in fact this combination results in the second player always being able to - worse than our first attempt!

So we can now add another initial condition. We must pick cards that

① - Prevent the second player from creating a royal flush

② - Don't let the second player block the first player turning his initial hand into straight flushes

To satisfy ① the first player must take royals from all 4 suits. However taking aces, kings, queens or jacks means that the second player can, for example, take all four tens - which then means you haven't satisfied ②. So how about if the first player took all the tens on his first turn?

The 10s are royal cards, so this prevents his opponent from creating a royal flush. However, the first player is still open to do so. The second player is then faced with 2 options.

i) Make the highest possible straight flush from the remaining cards
ii) Block the first player from creating a royal flush by taking four royals from the four different suits

If he pursues i) the first player can just discard three tens and the extra card from his hand, and turn the remaining ten into a royal flush. As the tens have been discarded the second player can't take them, and so he can't make a royal flush to make the first player's one, so the first player wins.

If he pursues ii) he must take 4 royals, and so can only take at most one card below a ten away from the first player. This leaves at least 3 suits in which the first player can form a straight flush to the 10. And, as the second player can't get hold of a 10, he can at best then create a straight flush to the 9, again meaning the first player wins.

Thus we see that if the first player picks all four tens and any other card, he can always win. There are clearly 48 ways of doing this (having taken the 10s there are 48 remaining cards). There is an additional solution, which is slightly more complicated but works along the same principle, in which he takes three of the 10s, and then any of {A-9, K-9, Q-9, J-9, K-8, Q-8, J-8, Q-7, Q-6, J-6} in the fourth suit. There are 40 ways of doing this (4 ways of choosing which suit will be the odd one out, and 10 combinations in each), so in total there are $48 + 40 = 88$ winning hands.